What are Tilings and Tessellations and how are they used in Architecture?

Tilings and tessellations are an important area of mathematics because they can be manipulated for use in art and architecture. One artist in particular, MC Escher, a Dutch artist, incorporated many complex tessellations into his artwork.

Tilings and tessellations are used extensively in everyday items, especially in buildings and walls. They are part of an area of mathematics that often appears simple to understand. However, research and investigation show that tilings and tessellations are in fact complex.

What are Tilings and Tessellations?

A tessellation is any repeating pattern of symmetrical and interlocking shapes. Therefore tessellations must have no gaps or overlapping spaces. Tessellations are sometimes referred to as “tilings”. Strictly, however, the word tilings refers to a pattern of polygons (shapes with straight sides) only. Tessellations can be formed from regular and irregular polygons, making the patterns they produce yet more interesting.

Tessellations of squares, triangles and hexagons are the simplest and are frequently seen in everyday life, for example in chessboards and beehives [Figure 1]. Tessellating other polygons, particularly irregular ones, is more difficult, as discussed later on.

History of Tessellations

The Latin root of the word tessellations is tessellare, which means ‘to pave’ or ‘tessella’, which means a small, square stone. Tessellations have been found in many ancient civilizations across the world. They often have specific characteristics depending on where they are from. Tessellations have been traced all the way back to the Sumerian civilizations (around 4000 BC). Tessellations were used by the Greeks, as small quadrilaterals used in games and in making mosaics [Figure 2]. Muslim architecture shows evidence of tessellations and an example of this is the Alhambra Palace at Granada, in the south of Spain [Figure 4]. The Fatehpur Sikri [Figure 3] also shows tessellations used in architecture.

Nowadays tessellations are used in the floors, walls and ceilings of buildings. They are also used in art, designs for clothing, ceramics and stained glass windows.

Mauritus Cornelius Escher had no higher knowledge of mathematics, yet he contributed to the mathematics of tessellations significantly. He was the youngest son of a hydraulics engineer and was born in Leeuwarden in the Netherlands in 1898. At high school he was an indifferent student and the only part of school that
interested him was art. After leaving school, Escher briefly studied under an architect before leaving to study ‘decorative arts’. He then began using graphic techniques in his own sketches.

Although Escher had no deep knowledge of mathematics his meticulous research into tilings of the plane was extensive and showed thorough mathematical research. In fact he once said, ‘Although I am absolutely innocent of training or knowledge in the exact sciences, I often seem to have more in common with mathematicians than with my fellow artists.’ (The Graphic Work of MC Escher, New York, 1967, p.9). Mathematicians have been fascinated by Escher’s work [Figure 5].

The Alhambra Palace in Granada, Spain with its ceilings...
and walls covered with beautiful ornamentation, sparked Escher’s interest in tessellating patterns. During his lifetime Escher created and designed over 100 tessellated patterns, many of which are still admired today.

**Mathematics Behind Tessellations**

**Which Shapes can Tessellate and Why**

The simplest type of tessellation is formed from regular polygons. Regular tessellations are tessellations that are made up of only one kind of regular polygon. After some experimentation it can be deduced that equilateral triangles, squares and regular hexagons form regular tessellations. However, pentagons and heptagons cannot do this without leaving gaps or producing overlaps.

In order for a shape to tessellate, the interior angles must fill all of the space around a vertex (i.e., their angles must add up to 360°). It is therefore important for us to work out the interior angles of a shape. In order to do this we must know that the exterior angles of any polygon will add up to 360°. From this we can deduce the interior angles of the polygon and check to see if they are factors of 360. If they are, we know that they will completely fill the space around a point, called a vertex. A vertex is the point at which adjacent sides of polygons meet. If this space is filled, the shape will tessellate.

An algebraic formula summarizes this. The formula to work out the interior angle of a regular polygon is: 
\[ a = 180 - \frac{360}{n} \]
where \( a \) is the interior angle of the polygon and \( n \) is the number of sides of the polygon. This is because 360 divided by the number of sides of the polygon gives the exterior angle, and when the exterior angle is subtracted from 180, we get the interior angle of the polygon.

To determine how many polygons are needed to fill the space around a vertex and allow the polygon to tessellate, another formula is used: 
\[ k(n) = \frac{360}{a} = \frac{2n}{n-2} \]
where \( k(n) \) is the number of polygons needed.

Therefore, it can be deduced that regular polygons that can fill the space around a vertex can tessellate. In more mathematical terms, regular polygons with interior angles that are a factor of 360, can tessellate. Because of this, only regular polygons with 3, 4 or 6 sides – equilateral triangles, squares and regular hexagons – can perfectly fill 360° and tessellate by themselves.

**Symmetry and Transformations**

Symmetry is the process of taking a shape and through certain movements, matching it exactly to another shape. A tessellation is created through this, by repeating the same motion a number of times. For example, the straight section of a railway line forms a tessellation as it uses the simplest type of symmetry. The same shape is repeated, moving a same fixed distance in the same fixed direction each time a movement is made.
Hexagon – The six exterior angles must add to make 360°. Therefore each one must be (360/6) 60°. The interior angles are (180-60) 120°. 120 is a factor of 360 and so a hexagon will tessellate.

Square – The four exterior angles must add to make 360°. Therefore each one must be (360/4) 90°. The interior angles are (180-90) 90°. 90 is a factor of 360 and so a square will tessellate.

Pentagon – The five exterior angles must add to make 360°. Therefore each one must be (360/5) 72°. The interior angles are (180-72) 108°. 108 is not a factor of 360 and so a pentagon will not tessellate.

Hexagon – The six exterior angles must add to make 360°. Therefore each one must be (360/6) 60°. The interior angles are (180-60) 120°. 120 is a factor of 360 and so a hexagon will tessellate.

Heptagon – The seven exterior angles must add to make 360°. Therefore each one must be (360/7) 51.4°. The interior angles are (180-51.4) 128.6°. 128.6 is not a factor of 360 and so a heptagon will not tessellate.
The techniques of forming symmetry are called transformations. These include translations, rotations, reflections and glide reflections.

One of the simplest types of symmetry is translational symmetry. A translation is simply a vertical, horizontal or diagonal slide. The shape must be moved a certain magnitude in a certain direction.

If a shape is marked on a square grid, vectors can be used to state a translation. Vectors appear as two numbers inside brackets e.g. \( \begin{pmatrix} 5 \\ 4 \end{pmatrix} \). These numbers refer to the movement of the shape along the x and y axis. Therefore the vector \( \begin{pmatrix} 5 \\ 4 \end{pmatrix} \) tells us to move the shape five units along the x-axis and four units up the y-axis. Vectors can also have negative values, to show the shape is being moved in the opposite direction.

Another type of symmetry is rotational symmetry. This is where a shape is moved a certain number of degrees around a central point, called the centre of rotation. The amount that the shape is turned is called the angle of rotation.

Rotations are used in tessellations to make shapes fit together. For example, the trapeziums above, once rotated and translated, fit together to produce rectangles. These rectangles have interior angles of 90° and so four of them \( 4 \times 90 \) will completely fill the space around a vertex, producing a tessellation.

If a tessellation has rotational symmetry it means that the tessellation can be rotated a certain number of degrees (other than 360°) to produce the same tessellation.

The most familiar type of symmetry is reflective symmetry. Reflections occur across a line called an axis. The distance of a point from this axis must be the same in the reflection. Therefore corresponding
All of the above transformations are used to produce tessellations when the basic, repeating unit pattern is known. They are also used to make shapes fit into one another in a tessellation of non-regular polygons.

**Polygon Arrangement Around A Vertex**

In order to understand the polygon arrangement around a vertex, we need to be able to specify exactly which polygons are found around the particular vertex. To do this, each tessellation is given a code which shows the number of sides of each polygon around the vertex. A triangle, for example, is a 3-sided shape and would therefore have the number 3.

When writing the polygon configuration around a vertex, we begin with the shape with the least sides and proceed clockwise around the vertex.

Example:

- Pick a vertex.
- Find the polygon with the smallest number of sides.
- Write down the number of sides that it has.

Above are the regular tessellations of equilateral triangles, squares and regular hexagons. They contain only one type of shape and are often referred to as periodic tessellations or tilings.

**Semi-regular Tessellations and Tilings**

Semi-regular tessellations are non-periodic i.e., they

Points should be the same distance away from the axis of reflection.

The last type of symmetry is glide reflection. A glide reflection is a reflection and a translation combined together. It does not matter which of the transformations happens first. The shape that emerges as a result of a reflection and translation is simply called the glide reflection of the original figure. In order for a glide reflection to take place an axis is needed to perform the reflection, and magnitude and direction are needed to perform the translation.

In this example of a glide reflection, the reflection is performed first and the translation is performed second.

The tessellation on the right has reflective symmetry.
contain more than one type of polygon. In a semi-regular tessellation the arrangement of polygons around a vertex must be the same at all vertices. However, only certain polygon arrangements will form semi-regular tessellations. This is because the interior angles of the polygons must fill 360°.

If two convex polygons met at a vertex, then the sum of their interior angles must be 360°. This would assume that one of the angles was 180° or more. This is impossible, however, leading to the conclusion that there must be at least 3 different types of polygon around a vertex.

In conclusion, semi-regular tessellations are more complex tessellations, with the same configuration of at least 3 polygons at each vertex.

Examples of semi-regular tessellations:
Other types of tessellation are demi-regular tessellations. These are similar to semi-regular tessellations, except there can be more than one different polygon arrangement at a vertex.

**Tilings and Tessellations of Non-Regular Polygons**
Non-regular polygons are those in which the interior angles are not the same, nor are the sides of the polygon of equal length. Non-regular polygons are slightly more difficult to tessellate, as they first require various transformations before they fit into a shape that will tessellate automatically. Non-regular tessellations are those in which there is no restriction on the order of the polygons around vertices.

All triangles and quadrilaterals will tessellate, but not all pentagons and hexagons will. When tessellating, we must remember that we are trying to find a polygon arrangement that will fill the 360° around a vertex.

**Triangles**
All interior angles of all triangles, whether equilateral, isosceles or scalene, will add up to 180°. Therefore we can fill the space around a vertex, if we use two of each of the angles of the triangle.
There are many different ways to arrange the angles of a triangle around a point to fill 360 degrees. Note that around each vertex, there are two copies of angle 1, two copies of angle 2, and two copies of angle 3.

The above examples will not tessellate because the sides are not lined up. If we line the sides up, however, we can tessellate any type of triangle. As the two adjacent angles add up to make 180 degrees, we can translate the two angles to form a straight line, along the tops of the pentagons. As the remaining three angles sum to 360 degrees, they can be rotated and translated to fill the space around a vertex.

There is another method that can be used to tessellate any type of triangle. This method is fairly similar to the one described above, except that after three triangles are arranged to form a 180° angle, the pattern is reflected, producing a slightly different tessellation.

**Quadrilaterals**
The method used to tessellate quadrilaterals is very similar to the method of tessellating triangles. We must first take into account that the interior angles of a quadrilateral add up to 360°. Therefore, we only need one of each angle to fill the space around a vertex. Again there are many possibilities that will not tessellate. We need therefore need to line up the sides to create a pattern that will tessellate.

**Pentagons**
Pentagons, as seen before, become much more difficult to tessellate because the sum of the interior angles of a pentagon is 540°, not 360°. Therefore, the angles will not fill the space around a vertex. However it is possible to tessellate pentagons, under certain conditions. If we think of the interior angles of a pentagon as 180° + 360°, rather than 540°, we can tessellate some pentagons.

There are three techniques of tessellating a pentagon. The first method requires the pentagon to contain 2 adjacent angles that add up to 180°. This method uses translations, rotations and reflections to extend the pentagon into a tessellation.

As the two adjacent angles add up to make 180°, we can translate the two angles to form a straight line, along the tops of the pentagons. As the remaining three angles sum to 360°, they can be rotated and translated to fill the space around a vertex.

In order to form a tessellation, the pattern can then be translated or reflected.

The second method requires the pentagon to contain two non-adjacent right angles, as well as two pairs of sides of equal length (as shown in the diagram).

This method ensures that all of the right angles meet at a vertex, in order to fill the 360°. This method uses multiple reflections, and takes advantage of the sides of equal length, to tessellate the pentagon.
The final technique of tessellating a pentagon begins with a triangle. One side of this triangle is broken up and two new sides are created from this, as shown in the diagram.

This new pentagon is then rotated around the midpoint, producing a parallelogram. This can be expanded as a quadrilateral (see earlier section) to form a tessellation.

Hexagons and other non-regular polygons can also form tessellations after various reflections, rotations and translations, which produce new shapes that can easily form tessellations.

The mathematics behind tessellations requires the space around a vertex to be filled completely without gaps or overlaps and in more complex shapes (pentagons etc.), this can be achieved by manipulating the shape.

Tessellations in Nature

Tessellations can be found everywhere we look. Nature contains many different tessellations, some being arrangements of polygons. Figure 6 shows some examples of tessellations found in nature.

Tessellations in Architecture

Tessellations are used extensively in architecture, both two-dimensional and three-dimensional. Tessellations are easy to use in architecture, especially in two-dimensional, because even the simplest repeating pattern can look astonishing when it covers a large area. Tessellations are used in Islamic architecture, for example the Taj Mahal [Figure 7], Agra and Fatehpur Sikri (a palace in India) [Figure 7]. Tessellations were also used by the Greeks and Romans in mosaics.

The top pictures in Figure 8 are of Hagia Sophia in Istanbul. They are examples of Islamic tessellations and tilings. The dome uses a three-dimensional tessellation of rectangles to cover it. There are further

Figure 6: Turtle shell, honeycomb, pineapple, spider web, giraffe markings and fish scales-examples of tessellations in nature
Figure 7: (Left to right) Mosaic at Chedworth Roman villa, decorative panel at Fatehpur Sikri, Patterns on the Taj Mahal

Figure 8: (Above) Hagia Sophia, Istanbul and (below) the Blue Mosque, Istanbul

Figure 9: Federation Square, Melbourne – this building is made up of a range of right-angled triangles. As can be seen, it is a very complex structure and is made out of many different materials.

Figure 10: London City Hall – This building projects a 2-D tessellation onto a ring. Each ring of the building is then slightly displaced

Figure 11: Louvre, Paris – Again the Louvre projects a 2-D tessellation onto a 3-D structure. Each separate face of the pyramid contains a section of a 2-D tessellation. A rhombus is used as the main shape
Figure 12: London Swiss Re Building. This building looks like a giant gherkin. It was made using computer modelling and without the use of complex computer models, buildings like this would not be possible. The gherkin uses rhombus shapes, which are manipulated to curve around the shape of the building.

Figure 13: British Museum. Ceiling of Great Hall. The grand court of the British Museum, London contains this beautiful, glass, tessellated roof. It uses triangles to fill the curved roof. This three-dimensional tessellation, like the one above, is very complex as it is not simply a projection of a two-dimensional tessellation onto a three-dimensional object. The question that arises from studying such a complex structure is: How can we manipulate a triangular tessellation to form a curve?

examples of Islamic tiling inside the building.

The beautiful tilings shown in Figure 8 are more examples of Islamic tilings at the Blue Mosque, also in Istanbul. The tilings are extremely intricate and delicate, covering the walls and ceiling.

The above examples are all of two-dimensional tessellations in architecture. Three-dimensional tessellations have also been used in various buildings and some examples are given below. Instead of using rectangular building blocks, the basic shape of the building blocks has been manipulated to produce interesting shapes. This is a difficult principle to master, but there have been some attempts. Buildings that use three-dimensional tessellations, once completed, generally are elegant and eye-catchimg. However, extensive planning and the use of complex computer models are required.

Acknowledgments

The following diagrams were taken from the ‘Totally Tessellated’ website and were not the author’s own: The source of the figures are as in the biography section.

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Books

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“Penrose Tiles to Trapdoor Ciphers” by Martin Gardner
“Geometric Symmetry” by E.H. Lockwood and R.H. Macmillan
“The Golden Ratio” by Mario Livio
“The Changing Shape of Geometry” edited by Chris Pritchard
“Indra’s Pearls – The Vision of Felix Klein” by David Mumford, Caroline Series and David Wright.